CARL VON OSSIETZKY universität OLDENBURG

Phase Transitions of Disordered Traveling Salesperson Problems solved with Linear Programming and Cutting Planes

Hendrik Schawe Alexander K. Hartmann

March 9, 2016
Regensburg

Traveling Salesperson Problem

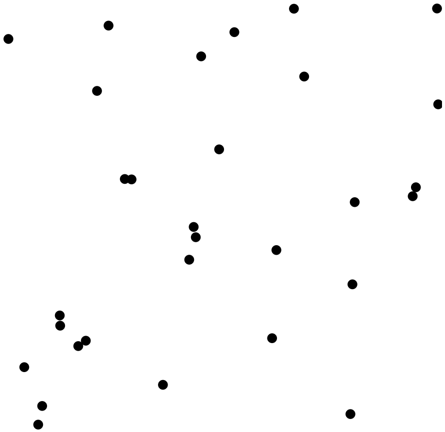
Linear Programming

Results

- Solution probability p
- Structural Properties

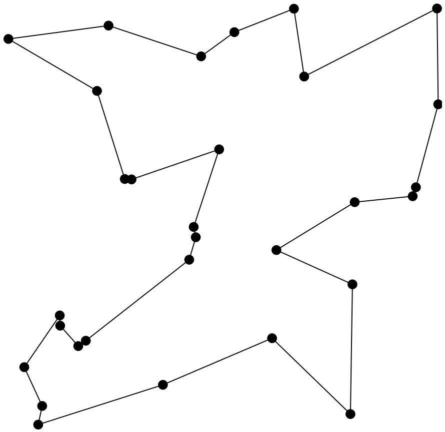
Traveling Salesperson Problem (TSP)

Given a set of cities V and their pairwise distances c_{ij}, what is the shortest tour visiting all cities and returning to the start?



Traveling Salesperson Problem (TSP)

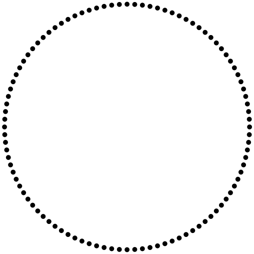
Given a set of cities V and their pairwise distances c_{ij}, what is the shortest tour visiting all cities and returning to the start?



Tunable Ensemble

Ensemble of disordered circles driven by the parameter σ

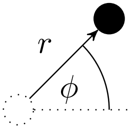
1. N cities on a circle
   with $R = N/2\pi$



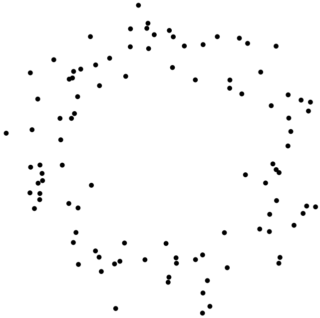
Tunable Ensemble

Ensemble of disordered circles driven by the parameter σ

1. N cities on a circle with $R = N/2\pi$
2. displace cities randomly



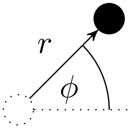
$r \in U[0, \sigma], \phi \in U[0, 2\pi)$



Tunable Ensemble

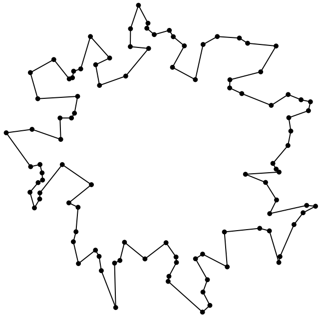
Ensemble of disordered circles driven by the parameter σ

1. N cities on a circle with $R = N/2\pi$
2. displace cities randomly



$r \in U[0, \sigma], \phi \in U[0, 2\pi)$

3. optimize the tour



Is there a phase transition easy circle $\rightarrow$ hard realization?

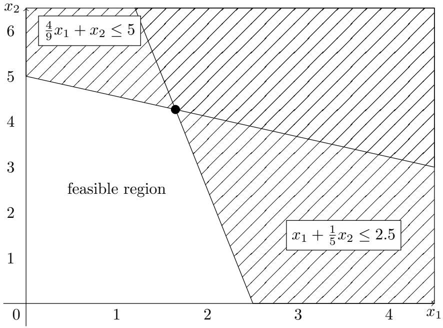
Linear Programming (LP)

$$\begin{aligned} \text{minimize} \quad & \mathbf{c}^T\mathbf{x} \\ \text{subject to} \quad & \mathbf{A}\mathbf{x} \le \mathbf{b} \\ & \mathbf{x} \in \mathbb{R}^N \end{aligned}$$

$$\mathbf{c} = \begin{pmatrix} -1 \\ -1 \end{pmatrix}$$

$$\mathbf{A} = \begin{pmatrix} \frac{4}{9} & 1 \\ 1 & \frac{1}{5} \end{pmatrix}$$

$$\mathbf{b} = \begin{pmatrix} 5 \\ 2.5 \end{pmatrix}$$



Linear Programming (LP)

$$\begin{aligned} \text{minimize} \quad & \mathbf{c}^T \mathbf{x} \\ \text{subject to} \quad & \mathbf{A}\mathbf{x} \leq \mathbf{b} \\ & \mathbf{x} \in \mathbb{R}^N \end{aligned}$$

- works outside the space of feasible solutions
- polynomial time
- can be used for combinatorial (integer) problems
  - is not always a valid solution
  - result valid $\rightarrow$ result optimal
  - yields at least a lower bound

LP formulation of the TSP

$x_{ij} = 1$ if i and j adjacent in tour

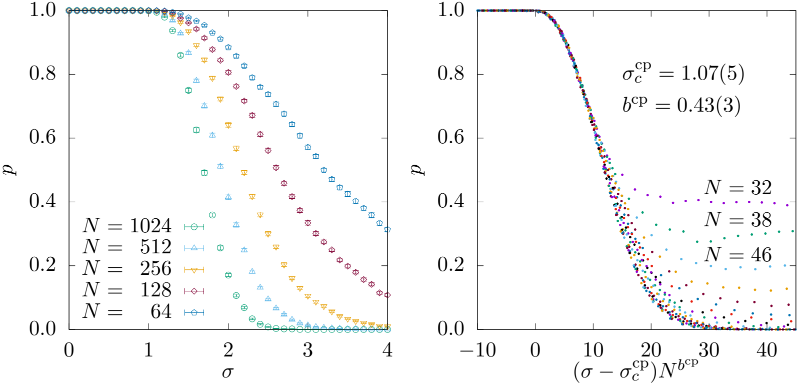
$$
\begin{aligned}
\text{minimize} \quad & \sum_i \sum_{j<i} c_{ij} x_{ij} \\
\text{subject to} \quad & \sum_j x_{ij} = 2 \qquad i = 1, 2, ..., N \\
& \sum_{i \in S, j \notin S} x_{ij} \geq 2 \qquad \forall S \subset V, S \neq \varnothing, S \neq V \quad \text{(SEC)} \\
& x_{ij} \in \{0, 1\}
\end{aligned}
$$

- $\forall S \subset V$ are exponentially many
  - add only violated (via cutting planes)
- x_{ij} are restricted to integer
  - relax it to $x_{ij} \in [0, 1]$ (may lead to infeasible solutions)

Dantzig, Fulkerson, Johnson, J. Oper. Res. Soc. Am., 2 (1954) 393

Solution probability p

Probability p that the SEC-relaxation is integer



Further transition at $\sigma \approx 0.5$ for weaker relaxation.

Schawe, Hartmann, EPL 113 (2016) 30004

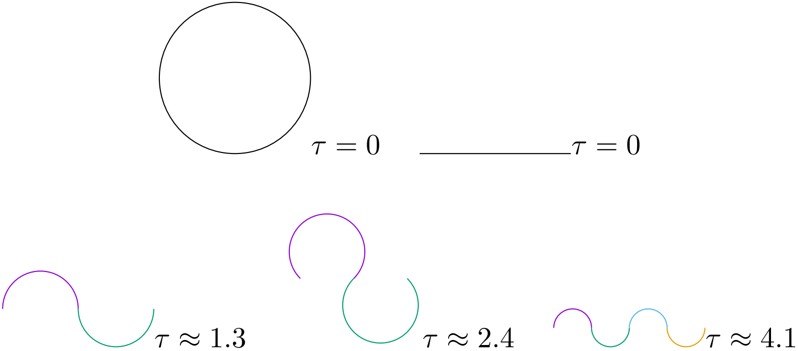
Structural Properties

Algorithmic phase transition
$\rightarrow$ search for physical properties that change

- solve them by branch-and-cut (exact $\rightarrow$ only small instances)

Tortuosity

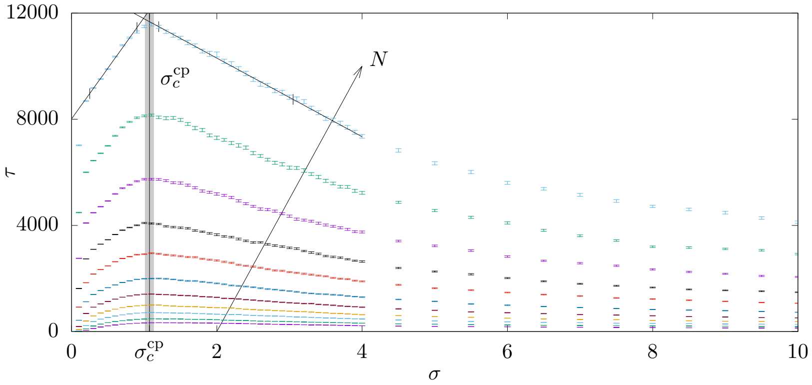
$$\tau = \frac{n-1}{L} \sum_{i=1}^{n} \left(\frac{L_i}{S_i} - 1 \right)$$



Grisan, Foracchia, Ruggeri, Proceedings of the 25th Annual International Conference of the IEEE Vol. 1 2003 pp. 866–869

Tortuosity

$$\tau = \frac{n-1}{L} \sum_{i=1}^{n} \left(\frac{L_i}{S_i} - 1 \right)$$



Universality

Same analysis with other ensembles or constraints

	σ_c	b
Degree relaxation	$\sigma_c^{\mathrm{lp}} = 0.51(4)$	$b^{\mathrm{lp}} = 0.29(6)$
SEC relaxation	$\sigma_c^{\mathrm{cp}} = 1.07(5)$	$b^{\mathrm{cp}} = 0.43(3)$
	$\sigma_c^{\tau} = 1.06(23)$	–
	$\sigma_c^{\mathrm{cp,g}} = 0.47(3)$	$b^{\mathrm{cp,g}} = 0.45(5)$
	$\sigma_c^{\tau\mathrm{,g}} = 0.44(8)$	–
	$\sigma_c^{\mathrm{cp,3}} = 1.18(8)$	$b^{\mathrm{cp,3}} = 0.40(4)$
fast Blossom rel.	$\sigma_c^{\mathrm{fb}} = 1.47(8)$	$b^{\mathrm{fb}} = 0.40(3)$

Summary

- linear programming to determine hardness
- three easy-hard transition points
- two structural properties changing at a transition