Large Deviations of Convex Hulls of Random Walks and Other Stochastic Models

Hendrik Schawe

19.03.2019

Large Deviations von

- Konvexe Hüllen von Random Walks
- Grundzustands-Energie eines Random-Energy Modells
- Zweifachzusammenhangs-Komponente von Zufallsgraphen
- Längste aufsteigende Teilfolge von Zufälligen Permutationen und Random Walks

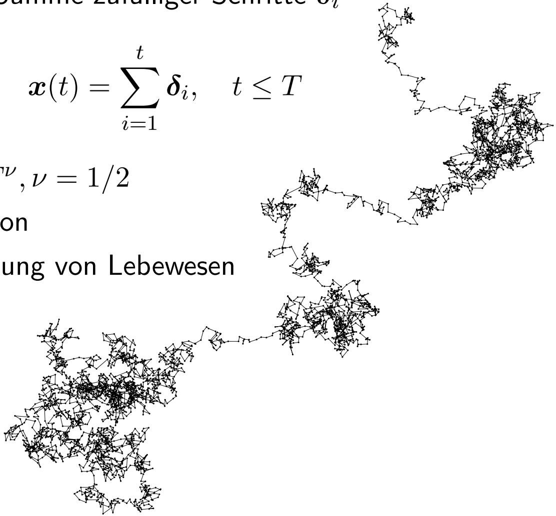
untersucht mittels Markov-Chain-Monte-Carlo Simulationen

Was sind Random-Walks?

- Position $\boldsymbol{x}(t)$ ist Summe zufälliger Schritte $\boldsymbol{\delta}_i$

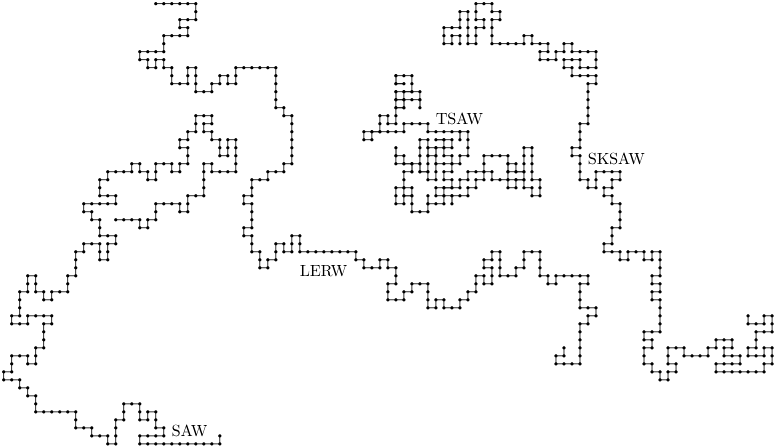
$$\boldsymbol{x}(t) = \sum_{i=1}^{t} \boldsymbol{\delta}_i, \quad t \leq T$$

- skaliert wie $r \propto T^\nu, \nu = 1/2$
- Modell für Diffusion
- Modell für Bewegung von Lebewesen



Was sind Self-Avoiding-Random-Walks?

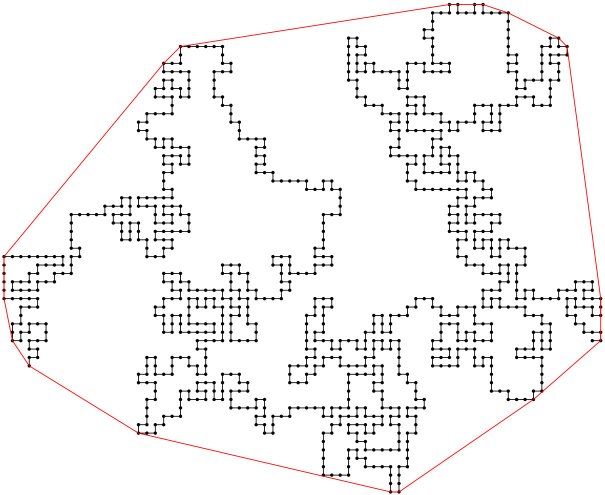
- Zusatzregeln zur Modellierung komplexerer Objekte
- Polymere: Zwei Teile können nicht das selbe Volumen belegen
- Wachstumsprozesse
- superdiffusiv in niedrigen Dimensionen ($\nu > 0.5$)



Smart-Kinetic-Self-Avoiding-Walk und konvexe Hülle

[animation]

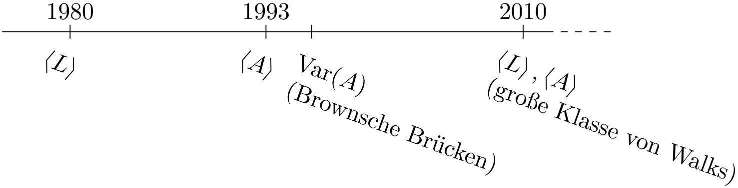
Smart-Kinetic-Self-Avoiding-Walk und konvexe Hülle



[animation]

Kovexe Hüllen von Random Walks

- Revier von Lebewesen Randon-Furling, Majumdar, Comtet (2009)
- Krankheitsausbreitung Dumonteil, Majumdar, Rosso, Zoia (2013)
- grundlegendes Interesse seit ~ 40 Jahren



Letac, Takács (1980), Letac (1993), Goldman (1996), Majumdar, Comtet, Randon-Furling (2010), Eldan (2014), Claussen, Hartmann, Majumdar (2015), Kabluchko, Zaporozhets (2016), Schawe, Hartmann, Majumdar (2017), Schawe, Hartmann, Majumdar (2018)

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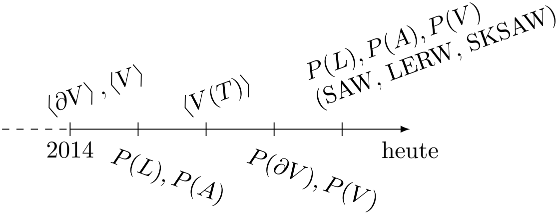

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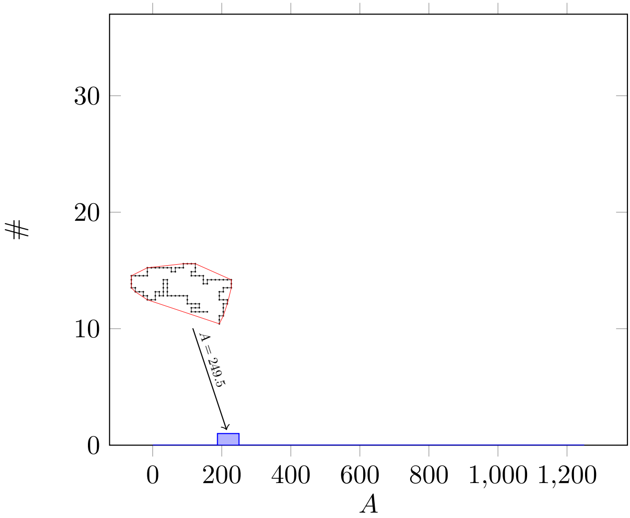
Verteilung der Fläche

$T = 100$



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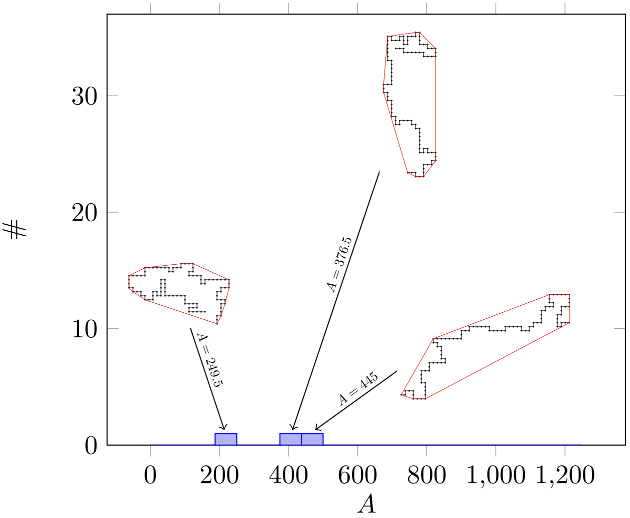

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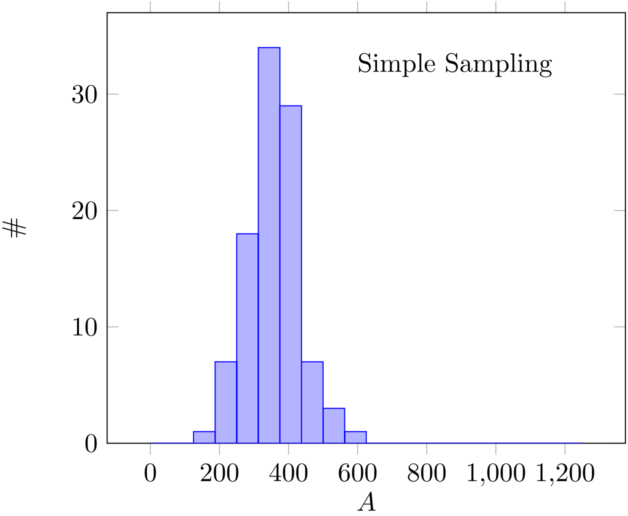

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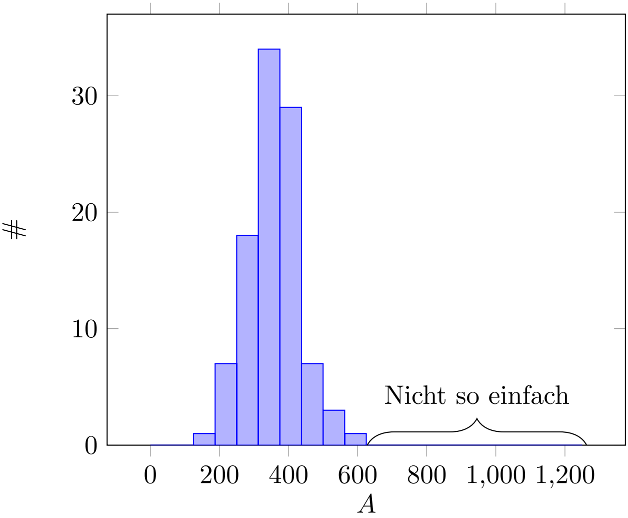
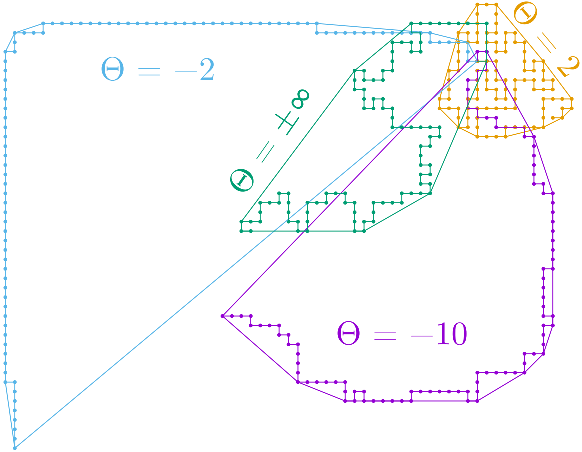


Large Deviation Simulation

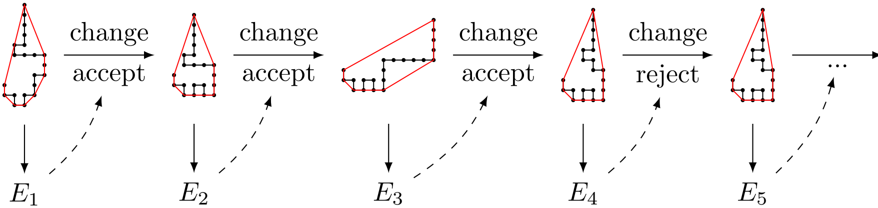
- Behandle Modell wie kanonisches System $\sim e^{-E/\Theta}$
- betrachte Fläche als Energie $A \equiv E$
- künstliche Temperatur Θ



Metropolis Algorithmus

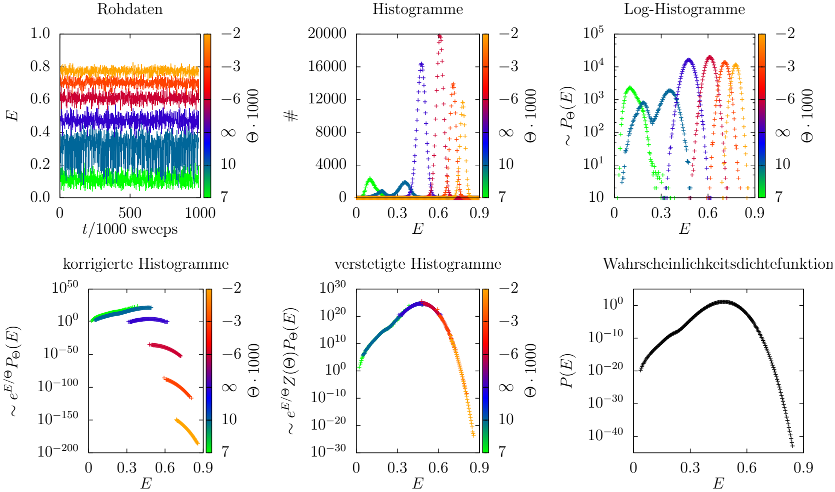
- Markov Kette von Zuständen = Realisierung von RWs
- akzeptiere kleine Änderung mit Wahrscheinlichkeit

$$p_{\mathrm{acc}} = \min\left\{1, e^{-\Delta E/\Theta}\right\}$$



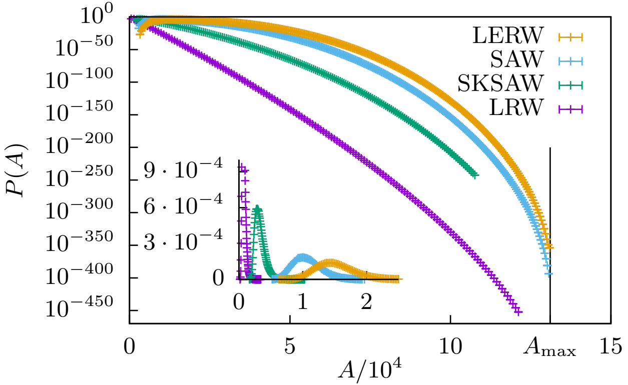
Metropolis et al., 1954

Large Deviation Simulation



Kovexe Hüllen von Random Walks

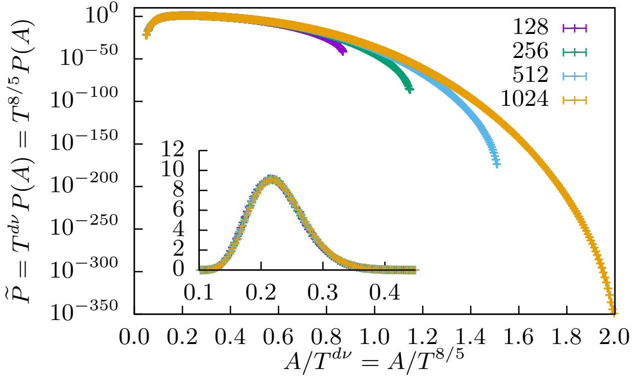
Verteilungen über alle möglichen Werte



$T = 1024$

Kovexe Hüllen von Random Walks

Mittelwerte skalieren wie $T^{d\nu}$. Verteilung auch?



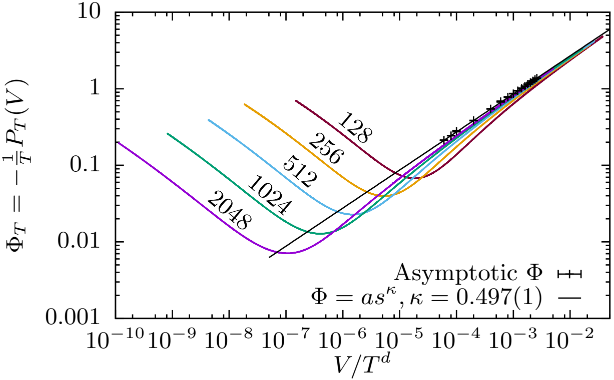
LERW, $\nu = 4/5$, $d = 2$

Exkurs: Large-Deviation-Theorie

Large-Deviation-Prinzip:
Für $T \to \infty$ ist die ganze Verteilung gegeben als Ratenfunktion:

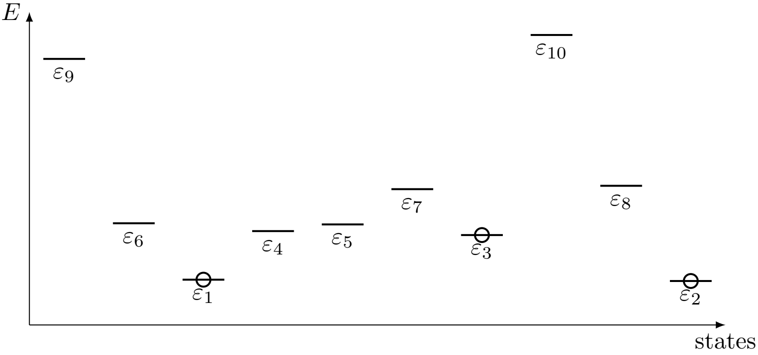
$$P_T(V) = e^{-T\Phi(V)+o(T)}$$

Ratenfunktion sollte wie $\Phi \propto V^\kappa, \kappa = \frac{1}{d(1-\nu)}$ gehen. Stimmt das?



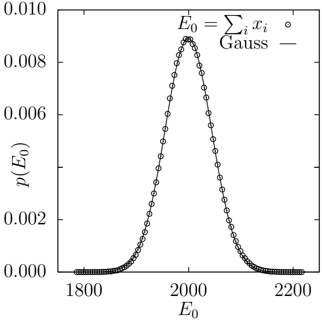
Grundzustandsverteilung eines Random-Energy-Modells

- N Energieniveaus ε_i, $(\varepsilon_1 \leq \varepsilon_2 \leq .. \leq \varepsilon_N)$
- unabhängig, identisch aus Verteilung $p(\varepsilon), \varepsilon \geq 0$
- K Fermionen
- Grundzustandsenergie $E_0 = \sum_{i=1}^{K} \varepsilon_i$
- ähnlich verallgemeinertem Spinglas-Modell Derrida (1980)

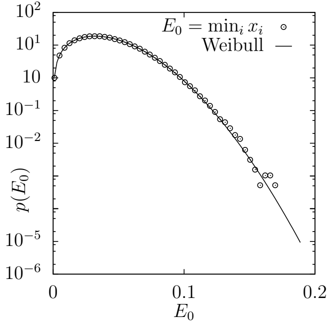


Extrema

$K = N$: Zentraler Grenzwertsatz: Gauß



$K = 1$: Extremwerttheorie: Weibull



Was ist die Verteilung für beliebige K?

Ergebnis von G. Schehr und S. Majumdar

Ausgehend von

$$P(\varepsilon_1, \cdots, \varepsilon_K) = \frac{\Gamma(N+1)}{\Gamma(N-K+1)} \prod_{i=1}^{K} p(\varepsilon_i) \prod_{i=2}^{K} \Theta(\varepsilon_i - \varepsilon_{i-1}) \left[\int_{\varepsilon_K}^{\infty} p(u)\, \mathrm{d}u \right]^{N-K}$$

wurde das Skalenverhalten für $N \to \infty$ hergeleitet

$$P_{K,N}(E_0) \approx b\, N^{\frac{1}{\alpha+1}} F_K^{(\alpha)} \left(b\, N^{\frac{1}{\alpha+1}}\, E_0 \right)$$

mit explizitem Ausdruck für

$$\int_0^{\infty} F_K^{(\alpha)}(z) e^{-\lambda\, z}\, \mathrm{d}z = \frac{(\alpha+1)^K}{\Gamma(K)\lambda^{(\alpha+1)(K-1)}} \int_0^{\infty} x^{\alpha} e^{-\lambda\, x - x^{\alpha+1}} \left[\gamma(\alpha+1, \lambda\, x)\right]^{K-1}\, \mathrm{d}x$$

Universell mit zwei Parametern

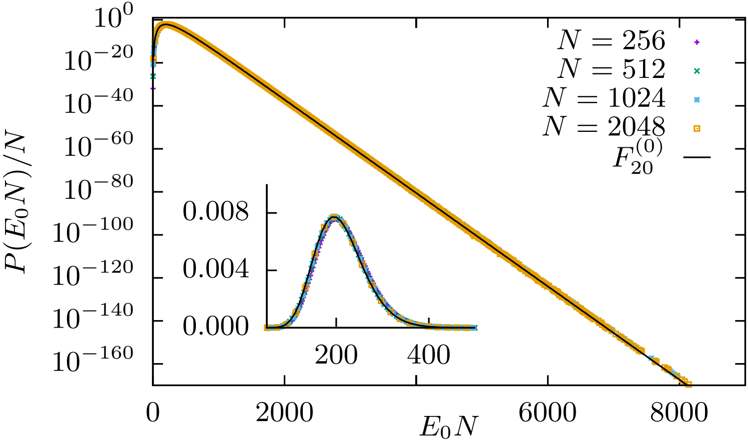
$$p(\varepsilon) \overset{\varepsilon \to 0}{\approx} B\varepsilon^{\alpha}$$

$$b = (B/(\alpha+1))^{1/(\alpha+1)}$$

Numerische Ergebnisse

Exponentialverteilte ε ($\alpha = 0, B = 1$), $K = 20$

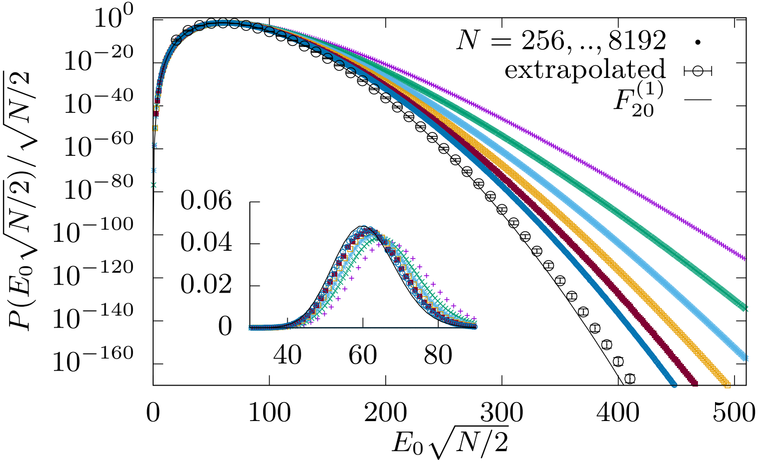
$$p(\varepsilon) = e^{-\varepsilon}, \varepsilon > 0$$



Numerische Ergebnisse

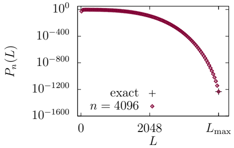
Erlangverteilte ε ($\alpha = 1, B = 1$), $K = 20$

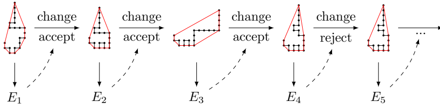
$$p(\varepsilon) = \varepsilon e^{-\varepsilon}, \varepsilon > 0$$

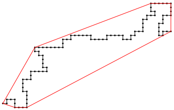


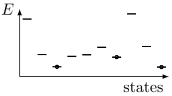
Zusammenfassung

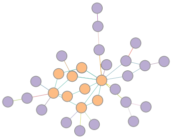
- Untersuchung von Verteilungen inklusive der Enden
- mittels Markov-Chain-Monte-Carlo
- für sehr unterschiedliche Modelle
- zur Untersuchung von Skalenverhalten
- und Ratenfunktionen

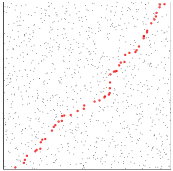










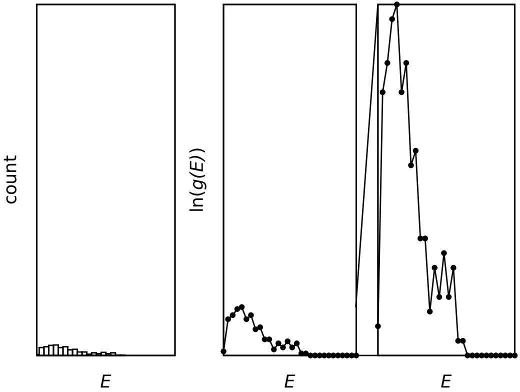


Wang-Landau

Ähnlich Metropolis, aber Akzeptanz mit

$$p_{\mathrm{acc}}(c_i \to c') = \min\left\{1, \frac{g(E(c_i))}{g(E(c'))}\right\}$$

$g(E)$ wird während der Simulation angepasst.



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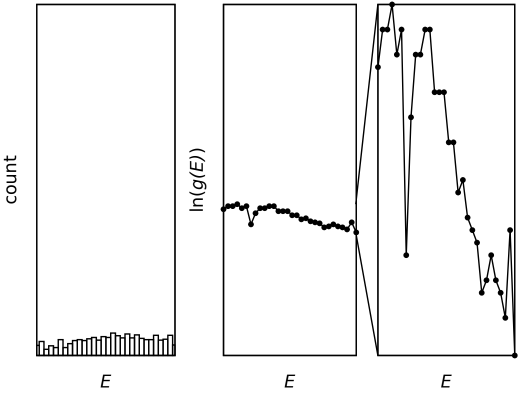

Hendrik Schawe

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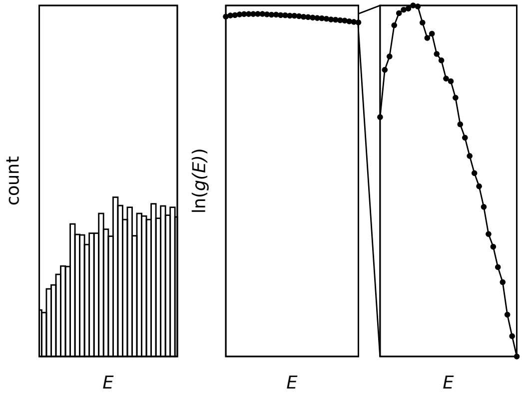

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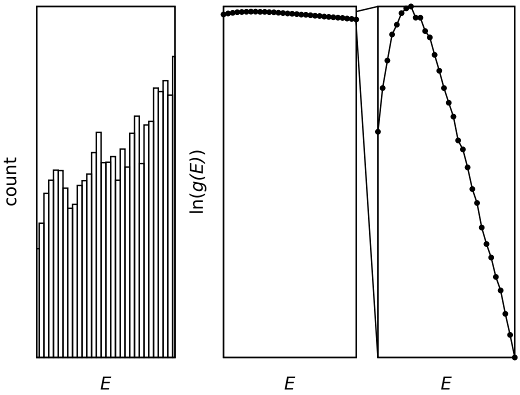

 Hendrik Schawe

Quickhull

Quickhull

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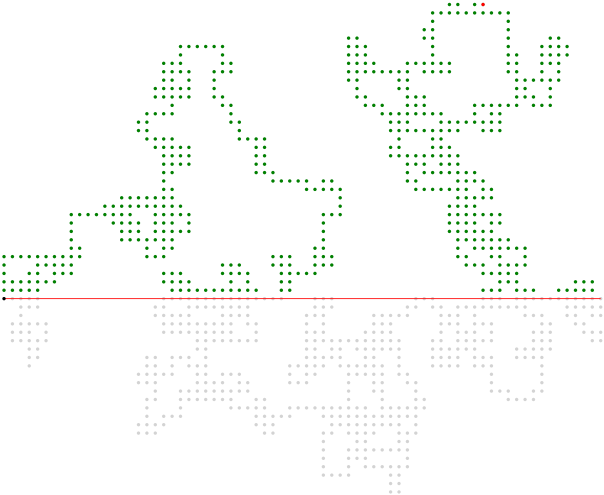

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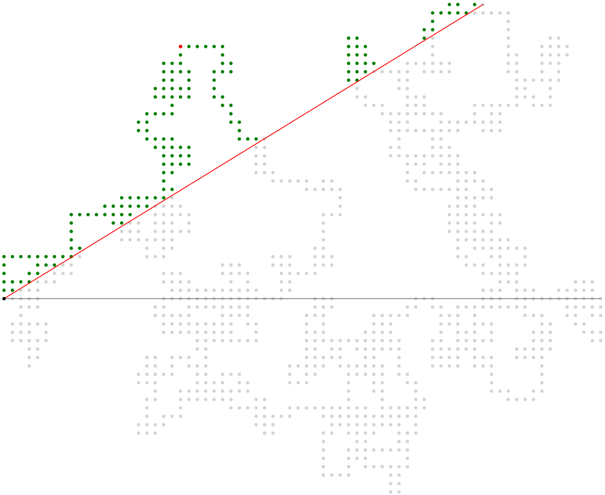

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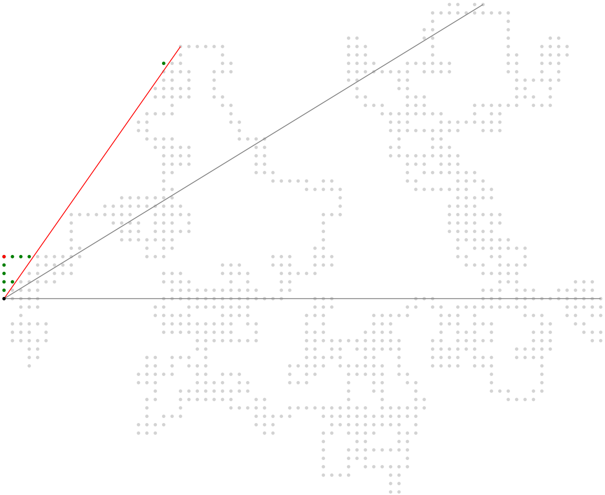

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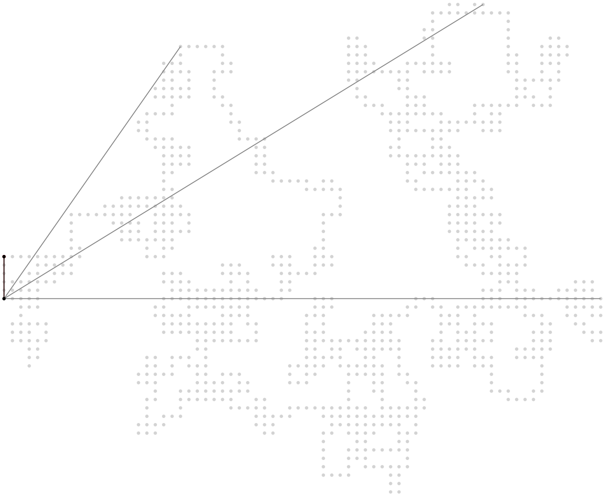

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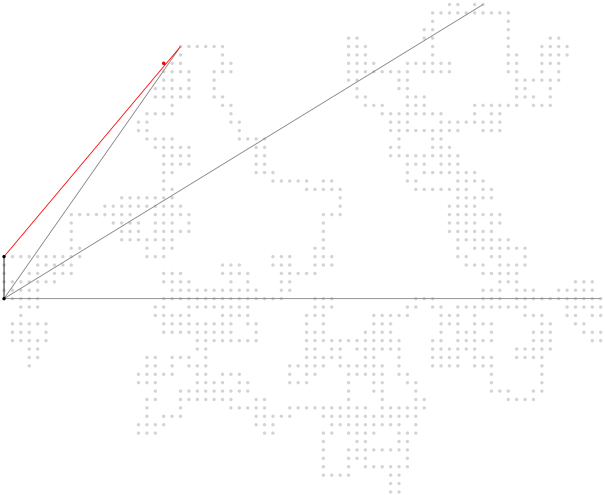

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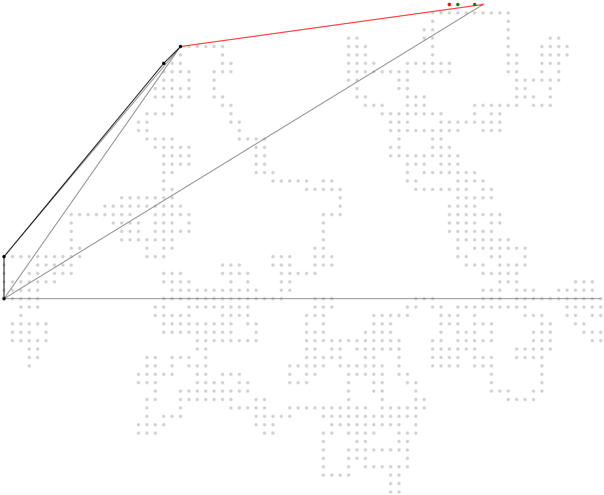

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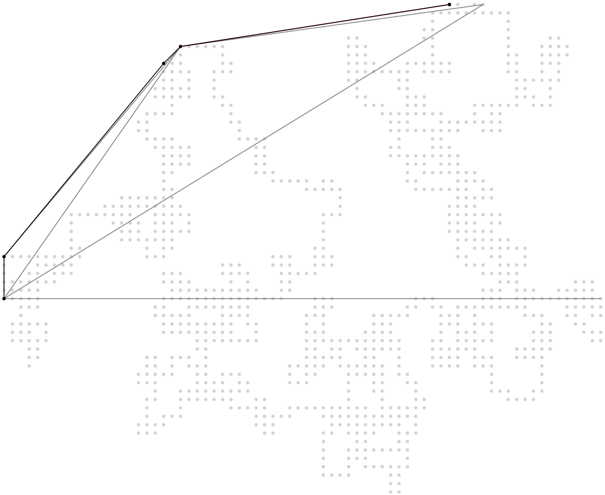

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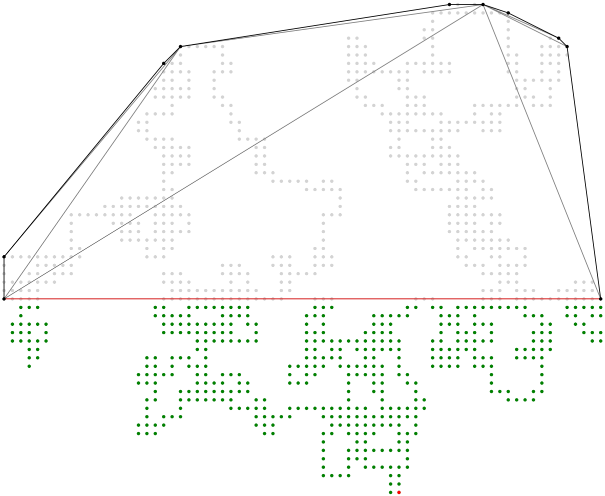

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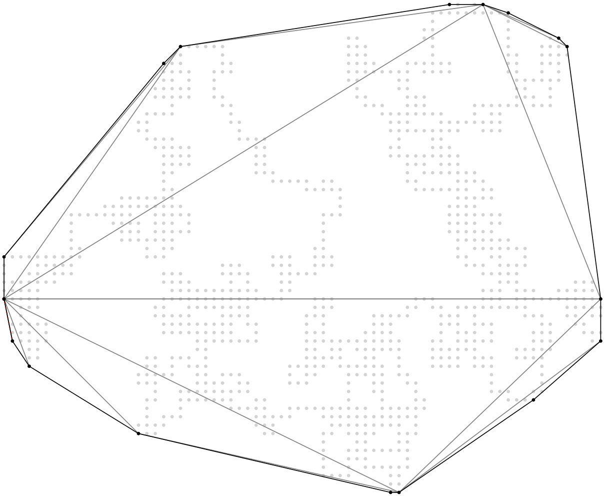

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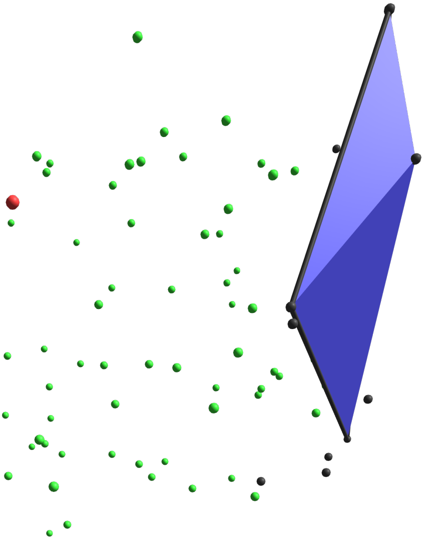

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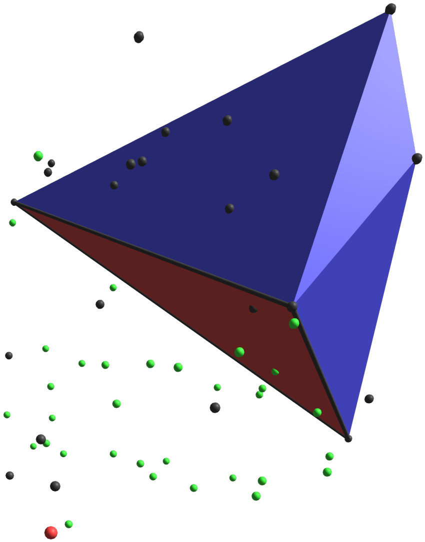

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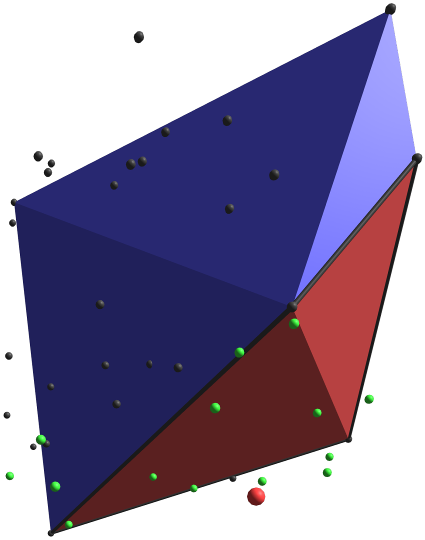

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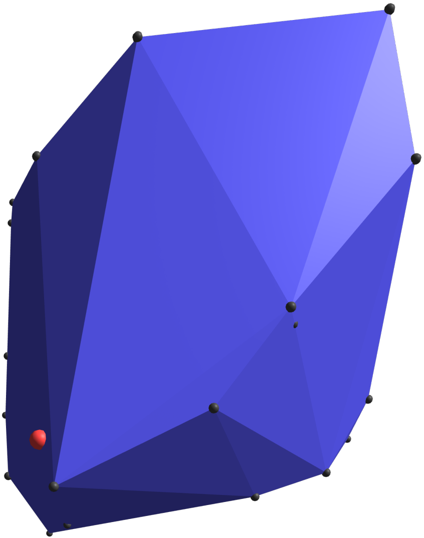
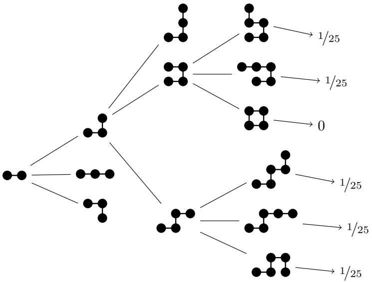
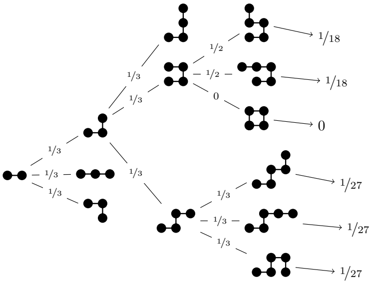


SAW vs. SKSAW

SAW



SKSAW



Warum sollte die Ratenfunktion so aussehen?

LDP: $P_T = \exp(-T\Phi(S/T^d))$
Ansatz: $\Phi \propto V^\kappa$
Annahme: Skalenform $P_T(V) = T^{-d\nu}\widetilde{P}(S/T^{d\nu})$ existiert
Argument: Wenn lhs $f(S/T^{d\nu})$, dann rhs auch $f(S/T^{d\nu})$

$$\begin{aligned} T^{-d\nu}\widetilde{P}(ST^{-d\nu}) &\approx \exp(-T\Phi(S/T^d)) \\ &\propto \exp(-T(S/T^d)^\kappa) \\ &= \exp(-T^{d\kappa-\nu d\kappa}(S/T^d)^\kappa) \\ &= \exp(-T^{-\nu d\kappa}S^\kappa) \\ &= \exp(-(S/T^{\nu d})^\kappa) \end{aligned}$$

Funktioniert nur mit:

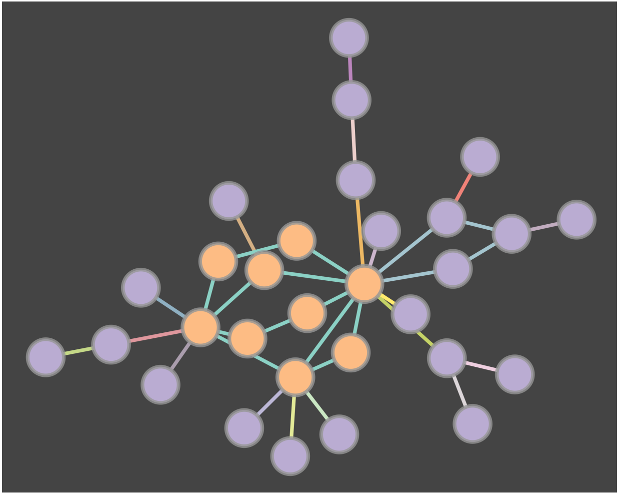
$$1 = d\kappa - vd\kappa \Rightarrow \kappa = \frac{1}{d(1-\nu)}$$

REM

$$P_{K,N}(E_0) = \int P(\varepsilon_1, \cdots, \varepsilon_K)\delta\left(E_0 - \sum_{i=1}^{K} \varepsilon_i\right) \prod_{i=1}^{K} \mathrm{d}\varepsilon_i$$

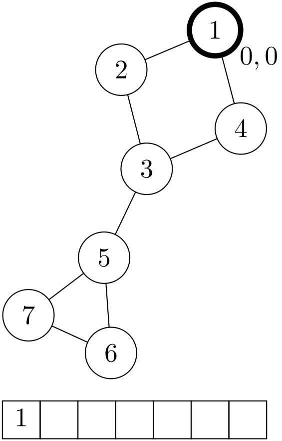
Laplacetransformation, Vereinfachungen und große N-Näherungen, und geeignete Skalierung führen auf N-unabhängige Form: F_K, deren Laplacetransformation bekannt ist.

Zweifachzusammenhangskomponenten



Algortihmus zur Komponentenfindung

Artikulationspunkt wenn *depth* $\leq$ *lowpoint* von Kind



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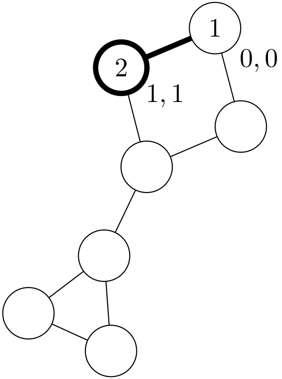

1	2					

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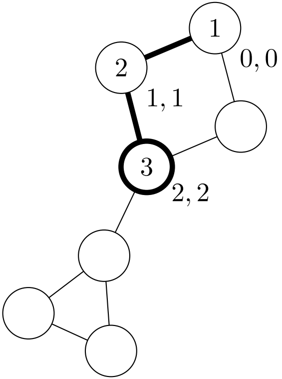

1	2	3				

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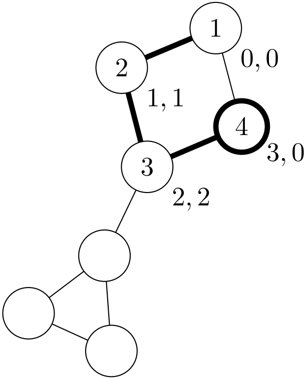
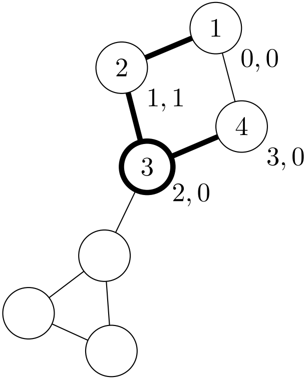


1	2	3	4			

Algortihmus zur Komponentenfindung

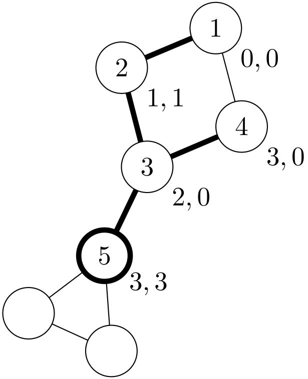
Artikulationspunkt wenn *depth* $\leq$ *lowpoint* von Kind



1	2	3				

Algortihmus zur Komponentenfindung

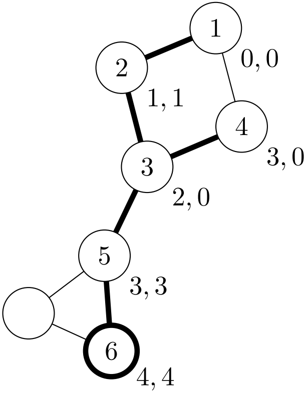
Artikulationspunkt wenn *depth* $\leq$ *lowpoint* von Kind



1	2	3	5			

Algortihmus zur Komponentenfindung

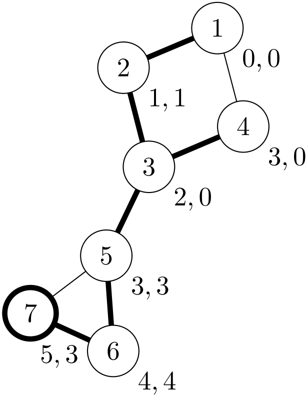
Artikulationspunkt wenn *depth* $\leq$ *lowpoint* von Kind



1	2	3	5	6		

Algortihmus zur Komponentenfindung

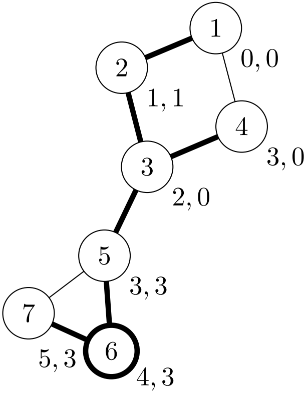
Artikulationspunkt wenn *depth* $\leq$ *lowpoint* von Kind



1	2	3	5	6	7	

Algortihmus zur Komponentenfindung

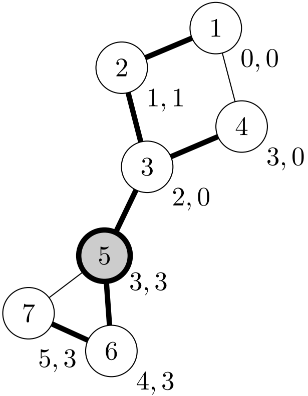
Artikulationspunkt wenn *depth* $\leq$ *lowpoint* von Kind



1	2	3	5	6		

Algortihmus zur Komponentenfindung

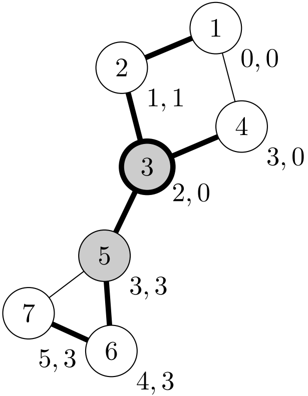
Artikulationspunkt wenn *depth* $\leq$ *lowpoint* von Kind



1	2	3	5			

Algortihmus zur Komponentenfindung

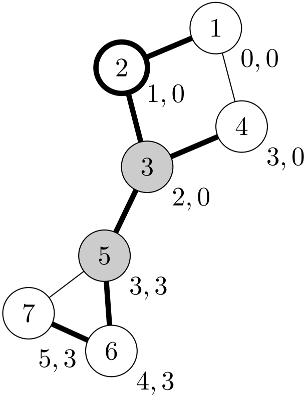
Artikulationspunkt wenn *depth* $\leq$ *lowpoint* von Kind



1	2	3				

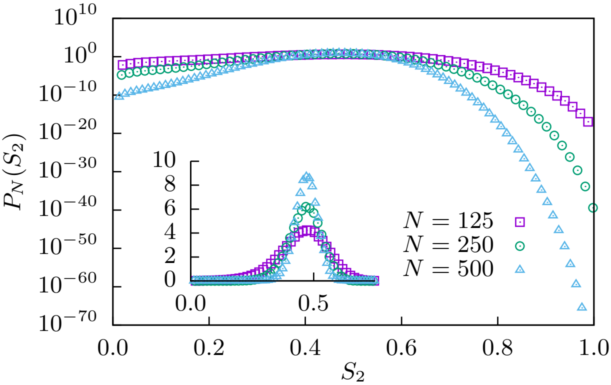
Algortihmus zur Komponentenfindung

Artikulationspunkt wenn *depth* $\leq$ *lowpoint* von Kind

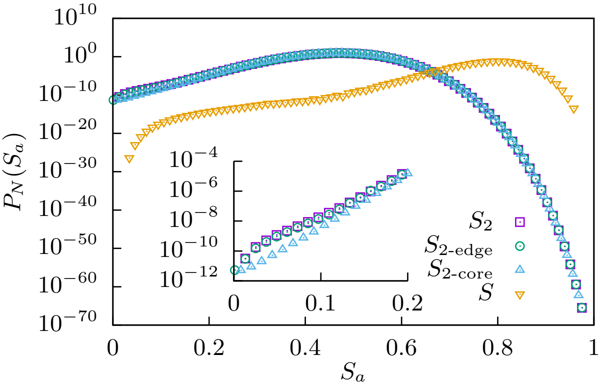


1	2					

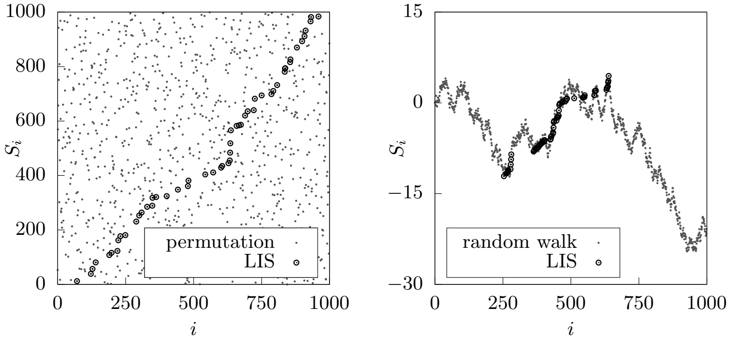
Verteilung Zweifachzusammenhangskomponentengröße



Verteilung verschiedene Komponenten



Längste aufsteigende Teilfolge

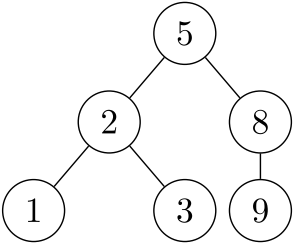


$$S = (\underline{3}, 9, \underline{4}, \overline{1}, \overline{2}, \underline{7}, \overline{6}, \overline{\underline{8}}, 0, 5)$$

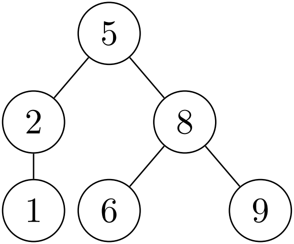
.

Datenstruktur für effiziente MCMC changes

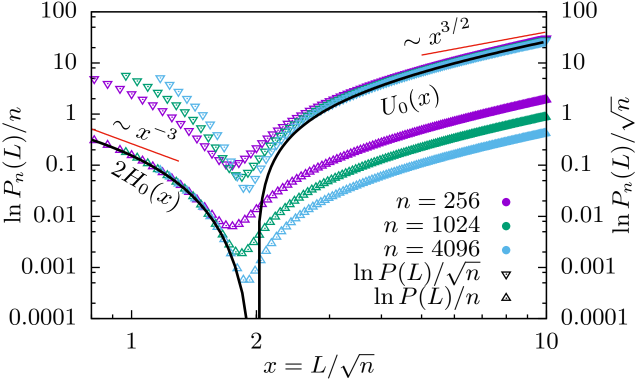
9	2	5	3	8	1



9	2	5	6	8	1



Ratenfunktionen, Zufallspermutation



Skalenverhalten, Random Walk